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Published in:
Mathematical Social Sciences

Publication date:
1986

Mathematical Social Sciences 12 (1986) 9–20
North-Holland

EXTENSIONS OF SOLUTION CONCEPTS BY MEANS OF MULTIPLICATIVE ε-TAX GAMES

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Communicated by F.W. Roush
Received 1 August 1985

Multiplicative ε-tax games are introduced that are obtained from a game by imposing a tax on any proper coalition. The imposed tax is proportional to the relative reward to the coalition for its formation from the individuals. Of particular interest is the balanced (quasi-balanced, respectively) ε-tax game for which the imposed tax is minimal. This least balanced (quasi-balanced) ε-tax game is used to extend the concept of the core (τ-value). In a similar way the stable core of a game is introduced by considering the least convex ε-tax game.

Key words: Game in characteristic function form; core; τ-value.

1. Introduction

A *cooperative n-person game in characteristic function form* is described by a finite nonempty set $N=\{1,2,\dots,n\}$ of *n players*, and a function v which assigns to any subset of N a real number. The real-valued function v on the family 2^N of subsets of N is called the *characteristic function* of the game and is required to satisfy $v(\emptyset)=0$. A subset S of the player set N is called a *coalition* and $v(S)$ is the *worth* of the coalition S in the game. A coalition S is said to be *proper* if $S\neq N,\emptyset$. In general, we identify a game by its characteristic function. The set of all n-person games is denoted by G^n.

Assuming that the grand coalition N is formed, the problem is to find a suitable distribution of the total earnings $v(N)$ among the players. Such a distribution is represented by a payoff vector $x=(x_1,x_2,\dots,x_n)\in\mathbb{R}^n$ satisfying the *efficiency condition* $\sum_{j=1}^n x_j=v(N)$. Here the ith coordinate x_i of the payoff vector x represents the payoff to player i. Usually it is also required that the *individual rationality condition* is satisfied, i.e. the payoff to any player is at least its single worth or equivalently $x_i\geq v(\{i\})$ for all $i\in N$. In view of this, we define for any n-person game v its *imputation set* $I(v)$ as the set consisting of all efficient, individually rational payoff vectors for the game v. Thus, for any $v\in G^n$

$$I(v):=\left\{x\in\mathbb{R}^n;\ \sum_{j=1}^n x_j=v(N)\text{ and }x_i\geq v(\{i\})\text{ for all }i\in N\right\}.$$

Clearly, $I(v) \neq \emptyset$ iff $v(N) \geq \sum_{j=1}^{n} v(\{j\})$. The set of all n-person games with a nonempty imputation set is denoted by I^n. Unless stated otherwise, we assume throughout the paper that $v \in I^n$.

The individual rationality condition can be strengthened to the *group rationality* condition, which requires that the payoff to any coalition is at least its worth. The *core* $C(v)$ of a game $v \in G^n$ is defined as the set consisting of all efficient, group rational payoff vectors for the game v. Thus, for any $v \in G^n$

$$C(v) := \left\{ x \in \mathbb{R}^n;\ \sum_{j=1}^{n} x_j = v(N) \text{ and } \sum_{j \in S} x_j \geq v(S) \text{ for all } S \subset N, S \neq \emptyset \right\}.$$

A game may have an empty core. It is well known that a game has a nonempty core if and only if the game is *balanced* (cf. Bondareva, 1963, and Shapley, 1967). The emptiness of the core is caused by the fact that the worth of one or more proper coalitions is 'too large' with respect to the worth of the grand coalition. There are many ways to lower the worths of proper coalitions, e.g. to impose a form of tax in all cases where a proper coalition is formed. The imposed tax may be a fixed positive real number ε, but the tax may also be proportional to the number of players in the coalition. These two *additive ways of imposing taxes* (i.e. considering for any proper coalition S the expressions $v(S) - \varepsilon$ and $v(S) - |S|\varepsilon$, respectively, instead of the worth $v(S)$) were introduced in Shapley and Shubik (1963, 1966) in order to generalize the concept of the core.

In this paper we pay special attention to a *multiplicative way of imposing taxes* by considering the multiplicative ε-tax games which are introduced in Section 2. Of particular interest are those ε-tax games which are obtained from the original game by imposing a 'minimal' tax.

In Section 3 we treat a generalization of the core with the aid of the least balanced ε-tax game, for which the imposed tax is minimal in the sense that a smaller imposed tax gives rise to an ε-tax game which is no longer balanced.

In a similar way we introduce in Section 5 the least quasi-balanced ε-tax game in order to extend the concept of the τ-value from the subclass of quasi-balanced n-person games (which is treated in Section 4) to the class I^n. In Section 6 we introduce for any game $v \in I^n$ its least stable core as the core of the least convex ε-tax game.

We conclude this section with some notation and definitions. Let $v, w \in G^n$, $c \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$ and θ be a permutation on N. Then the games $v + w$, $\alpha v + c$ and $\theta v \in G^n$ are given by

$$(v+w)(S) := v(S) + w(S), \qquad (\alpha v + c)(S) := \alpha v(S) + \sum_{j \in S} c_j$$

and

$$(\theta v)(\theta S) := v(S) \quad \text{for all } S \subset N,\ S \neq \emptyset.$$

With respect to this addition and multiplication, the set G^n of all n-person games is a $(2^n - 1)$-dimensional linear space. Furthermore, we write $v = w$ if $v(S) = w(S)$

for all $S \subset N$. Finally, the number of players in a coalition S is denoted by $|S|$, while Θ^n denotes the set of all permutations on N.

2. The bargaining game and multiplicative ε-tax games

In their generalization of the concept of the core, Shapley and Shubik (1963, 1966) made use of the 'shifted' games v_ε corresponding to any game $v \in G^n$ and any $\varepsilon \in \mathbb{R}$. The 'shifted' games are given by

$$v_\varepsilon(N) := v(N) \text{ and } v_\varepsilon(S) := v(S) - \varepsilon \quad \text{for all } S \subset N,\ S \neq N, \emptyset.$$

The underlying idea is to impose a tax of ε (or a bonus of $-\varepsilon$ if ε is negative) in all cases where a proper coalition is formed. Another possibility is to impose a tax of $|S|\varepsilon$ instead of ε for the formation of a proper coalition S. This second *additive way of imposing taxes* was also considered in Shapley and Shubik (1963, 1966).

In this section we introduce a certain *multiplicative way of imposing taxes.* The procedure is as follows. Consider a game $v \in I^n$, a proper coalition S and a real number ε with $0 \leq \varepsilon \leq 1$. If there is no cooperation between the members of S, then we suppose that any member i of S acts alone and earns his alternate worth $v(\{i\})$. In the case of cooperation, the coalition S is formed and its worth $v(S)$ can be seen as the total reward to the members of S for cooperation. So the relative reward to the coalition S for its formation from the individuals is given by $v(S) - \sum_{j \in S} v(\{j\})$. Now the idea is to impose a tax on the formation of S which is proportional to its relative reward, i.e. a tax of $\varepsilon[v(S) - \sum_{j \in S} v(\{j\})]$ is imposed (or a bonus of $-\varepsilon[v(S) - \sum_{j \in S} v(\{j\})]$ is given in case the 'relative reward' is negative). According to this procedure, the worth of the proper coalition S is obtained by adding the earnings of the members of S by noncooperation to the remaining relative reward to S for cooperation.

Definition. Let $n \geq 2$, $v \in I^n$ and $0 \leq \varepsilon \leq 1$. The *multiplicative ε-tax game* $v^\varepsilon \in G^n$ corresponding to v and ε is given by

$$\begin{aligned} v^\varepsilon(S) &:= \sum_{j \in S} v(\{j\}) + (1-\varepsilon)\left[v(S) - \sum_{j \in S} v(\{j\})\right] && \text{if } S \neq N, \emptyset, \\ &= v(S) && \text{if } S = N, \emptyset. \end{aligned}$$

If $\varepsilon = 0$, then no tax is imposed on the formation of any proper coalition and hence $v^0 = v$. If $\varepsilon = 1$, then the imposed tax on the formation of a proper coalition equals its total relative reward and hence

$$v^1(S) = \sum_{j \in S} v(\{j\}) \quad \text{if } S \neq N, \emptyset$$

and

$$v^1(S) = v(S) \qquad \text{if } S = N, \emptyset.$$

For any game $v \in I^n$, the game v^1 is called the corresponding *bargaining game* to v since in the game v^1 the grand coalition is the only multiperson coalition for which agreement on cooperation is interesting. Clearly, $I(v^1) = I(v) = C(v^1)$. The bargaining game v^1 usually belongs to any suitable chosen subclass of games.

If the game v is zero-normalized (i.e. $v(\{i\}) = 0$ for all $i \in N$), then the imposed tax on the formation of a proper coalition S is equal to $\varepsilon v(S)$ and hence $v^\varepsilon(S) = (1-\varepsilon)v(S)$. So the imposed tax is here multiplicative with respect to the game v itself.

Finally, we remark that in the case where the relative reward $v(S) - \sum_{j \in S} v(\{j\})$ to a proper coalition S is non-negative, then the imposed tax of $\varepsilon[v(S) - \sum_{j \in S} v(\{j\})]$ decreases whenever the tax-rate-factor ε decreases.

3. A generalization of the core by means of ε-tax games

Shapley and Shubik (1963, 1966) introduced the strong ε-cores as a generalization of the core. The strong ε-core $C_\varepsilon(v)$ of a game $v \in G^n$ is defined as the core of the 'shifted' game v_ε, which was mentioned at the beginning of Section 2. Clearly, $C_0(v) = C(v)$ and $C(v_\varepsilon) \subset C(v_\delta)$ whenever $\varepsilon \leq \delta$. The intersection of all nonempty strong ε-cores is called the least-core and was treated formally for the first time in Maschler, Peleg and Shapley (1979). The least-core can also be seen as the smallest nonempty strong ε-core. If the core is empty, then the least-core may be considered as revealing the 'latent' position of the core.

Now we treat a generalization of the core, which is based on the multiplicative ε-tax games v^ε corresponding to a game $v \in I^n$. Note that $I(v^\varepsilon) = I(v)$ since $v^\varepsilon(N) = v(N)$ and $v^\varepsilon(\{i\}) = v(\{i\})$ for all $i \in N$. Since $C(v^1) = I(v) \neq \emptyset$, the corresponding bargaining game v^1 is balanced, while the game $v (= v^0)$ itself is not necessarily balanced. For any $v \in I^n$ we define the critical value $\varepsilon^b(v)$ as the smallest non-negative ε for which the ε-tax game $v^\varepsilon = \varepsilon v^1 + (1-\varepsilon)v^0$ is balanced. Thus, for any $v \in I^n$

$$\varepsilon^b(v) := \min\{\varepsilon;\ \varepsilon \geq 0, C(v^\varepsilon) \neq \emptyset\}.$$

The number $\varepsilon^b(v)$ is well defined since the subclass B^n of balanced n-person games is a convex polyhedral cone. Clearly, $v \in B^n$ iff $\varepsilon^b(v) = 0$.

Definition. Let $v \in I^n$, where $n \geq 2$. The *least-tax-core* LTC(v) *of the game* v is defined as the core of the least balanced ε-tax game $v^{\varepsilon^b(v)} \in B^n$.

If the core is nonempty (i.e. if $\varepsilon^b(v) = 0$), then the least-tax-core coincides with the core. If the core is empty (i.e. if $\varepsilon^b(v) > 0$), then the least-tax-core may be considered as revealing the 'latent' position of the core. It follows from the next lemma that if the relative reward $v(S) - \sum_{j \in S} v(\{j\})$ to any coalition S for its formation from the individuals is indeed non-negative, then the least-tax-core of the game v can geometrically be obtained as the intersection of all nonempty cores of ε-tax games corresponding to the game v.

Lemma. *Let $n \geq 2$ and $v \in G^n$ such that $v(S) \geq \sum_{j \in S} v(\{j\})$ for all $S \subset N$, $S \neq \emptyset$. Then $C(v^\varepsilon) \subset C(v^\delta)$ whenever $0 \leq \varepsilon \leq \delta \leq 1$.*

The proof of the lemma is straightforward and is left to the reader. The least-tax-core is relative invariant under S-equivalence, i.e.

$$\mathrm{LTC}(\alpha v + c) = \alpha \mathrm{LTC}(v) + c \text{ whenever } v \in I^n,\ c \in \mathbb{R}^n \text{ and } \alpha > 0.$$

The validity of this invariance is based on the facts that $(\alpha v + c)^\varepsilon = \alpha v^\varepsilon + c$ and $\varepsilon^b(\alpha v + c) = \varepsilon^b(v)$ whenever $v \in I^n$, $c \in \mathbb{R}^n$ and $\alpha > 0$.

The least-core and the least-tax-core are compared in the next example.

Example. Let the 3-person game v be given by $v(\{i\}) = 0$ for all $i \in N$, $v(\{1,2\}) = 5$, $v(\{1,3\}) = 12$, $v(\{2,3\}) = 13$ and $v(N) = 27/2$. Then $C(v) = \emptyset$. Since the game v is zero-normalized, we have that the ε-tax game v^ε is given by $v^\varepsilon(N) = 27/2$ and $v^\varepsilon(S) = (1-\varepsilon)v(S)$ for all $S \neq N$. It turns out that the ε-tax game v^ε is balanced iff $\varepsilon \geq 1/10$, while the 'ε-shifted' game v_ε is balanced iff $\varepsilon \geq 1$. Furthermore, $C(v^{1/10}) = \{(18/10, 27/10, 90/10)\}$ while $C(v_1) = \{(3/2, 5/2, 19/2)\}$.

In Maschler, Peleg and Shapley (1979) it was shown that the strong ε-cores are very useful for the study of several solution concepts such as the (pre)kernel and the nucleolus. The relationship between certain solution concepts and the cores of ε-tax games is still open to investigation.

4. The τ-value of a quasi-balanced game

In this section we recall the definition and some properties of the τ-value for the quasi-balanced games. For that purpose we consider the next notions.

Definition. Let $v \in G^n$. The *upper vector* $b^v = (b_1^v, b_2^v, \ldots, b_n^v) \in \mathbb{R}^n$, the *gap function* $g^v : 2^N \to \mathbb{R}$ and the *concession vector* $\lambda^v = (\lambda_1^v, \lambda_2^v, \ldots, \lambda_n^v) \in \mathbb{R}^n$ of the game v are given by

$$b_i^v := v(N) - v(N - \{i\}) \quad \text{for all } i \in N,$$

$$g^v(S) := \sum_{j \in S} b_j^v - v(S) \quad \text{for all } S \subset N, S \neq \emptyset, g^v(\emptyset) := 0,$$

$$\lambda_i^v := \min_{S;\, i \in S} g^v(S) \quad \text{for all } i \in N.$$

The ith coordinate b_i^v of the upper vector b^v represents the marginal contribution of player i to the grand coalition in the game v. The term 'upper vector' is explained by the fact that the vector is an upper bound for the core, i.e.

$$x_i \leq b_i^v \quad \text{for all } x \in C(v) \text{ and all } i \in N. \tag{4.1}$$

For games v with a non-negative gap function (condition 1) the upper vector b^v can be seen as a utopian payoff vector, which is, however, generally not efficient. With respect to the utopian vector b^v, an efficient payoff vector is obtained whenever the grand coalition makes a concession amount of $g^v(N)$. Now the essential idea is that the contribution of any player i to the total concession amount $g^v(N)$ is proportional to his concession amount λ_i^v in such a way that his contribution to $g^v(N)$ is at most his concession amount λ_i^v (condition 2). Games which satisfy the conditions 1 and 2 are called quasi-balanced.

Definition. The subclass QB^n of *quasi-balanced n-person games* is given by

$$QB^n := \left\{ v \in G^n;\ g^v(S) \geq 0 \text{ for all } S \subset N \text{ and } \sum_{j=1}^{n} \lambda_j^v \geq g^v(N) \right\}.$$

Definition. The *τ-value* $\tau(v) \in \mathbb{R}^n$ of a quasi-balanced n-person game v is given by

$$\begin{aligned} \tau(v) &:= b^v && \text{if } g^v(N) = 0, \\ &= b^v - g^v(N) \left[\sum_{j=1}^{n} \lambda_j^v \right]^{-1} \lambda^v && \text{if } g^v(N) > 0. \end{aligned}$$

Our interest in the concession vector λ^v is due to the fact that the disagreement vector $b^v - \lambda^v$ is a lower bound for the core, i.e.

$$b_i^v - \lambda_i^v \leq x_i \quad \text{for all } x \in C(v) \text{ and all } i \in N. \tag{4.2}$$

In a straightforward way, (4.1) and (4.2) imply that games with a nonempty core (or equivalently balanced games) are quasi-balanced. Furthermore, the subclass QB^n is a full-dimensional polyhedral cone in G^n (cf. Tijs, 1981).

Geometrically the τ-value of a quasi-balanced game v coincides with the unique intersection point of the hyperplane of efficient payoff vectors for v with the straight line segment which has the points $b^v - \lambda^v$ and b^v as its end points (see Fig. 1). As such the τ-value of a quasi-balanced game was introduced in Tijs (1981).

For the proofs of (4.1) and (4.2), for the relationship between balancedness and quasi-balancedness, and for a detailed justification of the concession vector, we refer to Tijs (1981), Tijs and Driessen (1986) and Driessen and Tijs (1985). In the next theorem we recall several properties of the τ-value on QB^n. Here the set D^v represents the *dummy players in the game* v, i.e. for any $v \in G^n$

$$D^v := \{ i \in N;\ v(S \cup \{i\}) - v(S) = v(\{i\}) \text{ for all } S \subset N - \{i\} \}.$$

Theorem (cf. Tijs, 1981). *Let $v \in QB^n$. The τ-value $\tau(v)$ possesses the following properties:*

(i) *efficiency:* $\sum_{j=1}^{n} \tau_j(v) = v(N)$;
(ii) *individual rationality:* $\tau_i(v) \geq v(\{i\})$ *for all* $i \in N$;
(iii) *dummy player property:* $\tau_i(v) = v(\{i\})$ *for all* $i \in D^v$;
(iv) *symmetry:* $\tau_{\theta(i)}(\theta v) = \tau_i(v)$ *for all* $i \in N$ *and all* $\theta \in \Theta^n$;
(v) *relative invariance under S-equivalence:* $\tau(\alpha v + c) = \alpha \tau(v) + c$ *for all* $c \in \mathbb{R}^n$ *and all* $\alpha > 0$.

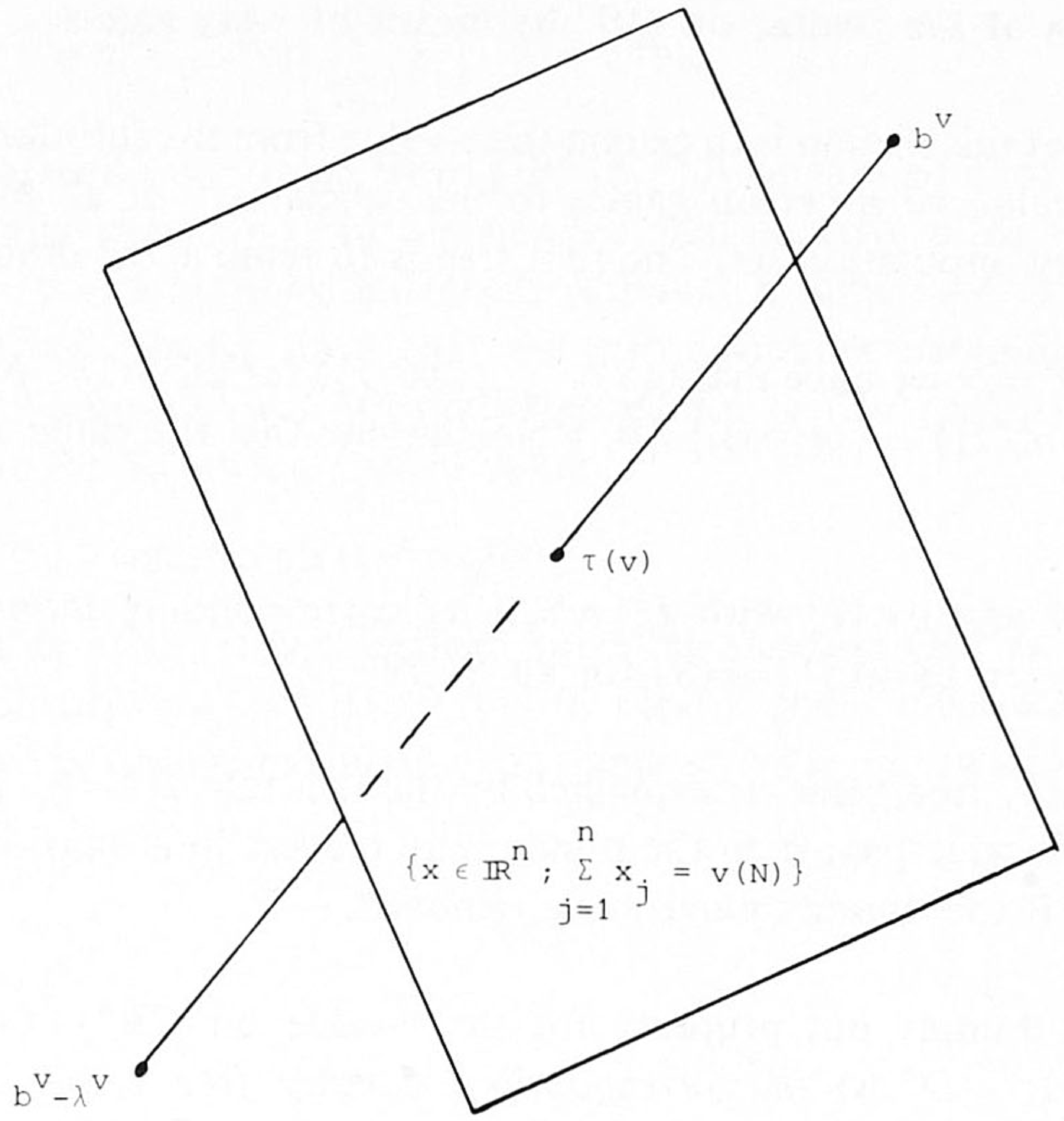

Fig. 1. The τ-value $\tau(v)$ of a game $v \in QB^n$.

An extension of the τ-value from QB^n to G^n was treated in Driessen and Tijs (1984) by considering the smallest non-negative ε for which the corresponding 'ε-shifted' game is quasi-balanced. However, that extension of the τ-value to G^n fails to possess the individual rationality and the dummy player property. In the next section we treat another extension of the τ-value by considering ε-tax games instead of 'ε-shifted' games. That second extension of the τ-value requires the nonemptiness of the imputation set and takes into account the presence of dummy players. Making use of the above properties for the τ-value on QB^n, we also show that the second extension of the τ-value possesses the same properties. In view of this, it is meaningful to recall the proofs of (ii) the individual rationality and (iii) the dummy player property of the τ-value on QB^n. Let $v \in QB^n$ and $i \in N$.

Finally, we remark that an extensive study on the τ-value of quasi-balanced games can be found in Driessen (1985).

ad (ii). By the definition of the τ-value we have that $\tau_i(v) \geq b_i^v - \lambda_i^v$. Since $\lambda_i^v \leq g^v(\{i\})$, it follows that $\tau_i(v) \geq b_i^v - g^v(\{i\}) = v(\{i\})$.

ad (iii). By the definition of the τ-value we have that $\tau_i(v) \leq b_i^v$. In the case where player i is a dummy player, it follows that $\tau_i(v) \leq b_i^v = v(N) - v(N - \{i\}) = v(\{i\})$ and hence $\tau_i(v) = v(\{i\})$ by using (ii).

5. An extension of the τ-value on QB^n by means of ε-tax games

The purpose of this section is to extend the τ-value from the full-dimensional cone QB^n of quasi-balanced n-person games to the subclass I^n of all n-person games with a nonempty imputation set. The first step is to remove the dummy players in a game $v \in I^n$.

In the case $D^v = N$ we have that $v(S) = \sum_{j \in S} v(\{j\})$ for all $S \subset N$, $S \neq \emptyset$ and hence $C(v) = \{(v(\{1\}), v(\{2\}), \ldots, v(\{n\}))\} \neq \emptyset$, which implies that the game v is also quasi-balanced.

Definition. For any $v \in G^n$ with $D^v \neq N, \emptyset$ its corresponding *dummy free game* $(N - D^v, w)$ is given by $w(S) := v(S)$ for all $S \subset N - D^v$.

The term 'dummy free game' is explained by the fact that $D^w = \emptyset$. The next result states that the τ-value payoff to the nondummy players in a quasi-balanced game is not affected if the dummy players are removed.

Theorem (The dummy out property for the τ-value on QB^n). *Let $v \in G^n$ with $D^v \neq N, \emptyset$ and $(N - D^v, w)$ its corresponding dummy free game. If v is quasi-balanced, then w is also quasi-balanced and $\tau_i(w) = \tau_i(v)$ for all $i \in N - D^v$.*

Proof. For $i \in D^v$ and $j \in N - D^v$ we have that $b_i^v = v(N) - v(N - \{i\}) = v(\{i\})$ while

$$\begin{aligned} b_j^w &= w(N - D^v) - w((N - D^v) - \{j\}) = v(N - D^v) - v((N - D^v) - \{j\}) \\ &= v(N) - v(N - \{j\}) = b_j^v. \end{aligned}$$

From this it follows that for all $S \subset N$, $S \neq \emptyset$,

$$\begin{aligned} g^v(S) &= \sum_{j \in S} b_j^v - v(S) = \sum_{j \in S - D^v} b_j^v - v(S - D^v) \\ &= \sum_{j \in S - D^v} b_j^w - w(S - D^v) = g^w(S - D^v). \end{aligned}$$

This implies that $\lambda_i^w = \lambda_i^v$ for all $i \in N - D^v$.

Let $v \in QB^n$. Then $\lambda_i^v = 0$ for all $i \in D^v$ since $v \in QB^n$ and $i \in D^v$ imply that $0 \leq \lambda_i^v \leq g^v(\{i\}) = b_i^v - v(\{i\}) = 0$. Hence,

$$\sum_{j \in N - D^v} \lambda_j^w = \sum_{j \in N} \lambda_j^v.$$

Note that $g^v(N) = g^w(N - D^v)$. Now it follows that w is also quasi-balanced. Moreover, in the case $g^v(N) = 0$ we have that $\tau_i(w) = b_i^w = b_i^v = \tau_i(v)$ for all $i \in N - D^v$, while in the case $g^v(N) > 0$ we have for all $i \in N - D^v$

$$\tau_i(w) = b_i^w - g^w(N - D^v)\left[\sum_{j \in N - D^v} \lambda_j^w\right]^{-1} \lambda_i^w$$

$$= b_i^v - g^v(N)\left[\sum_{j\in N} \lambda_j^v\right]^{-1} \lambda_i^v = \tau_i(v). \qquad \square$$

In order to extend the τ-value from QB^n to I^n by means of ε-tax games, we are interested for any game $v \in I^n$ in the smallest non-negative ε for which the ε-tax game $v^\varepsilon = \varepsilon v^1 + (1-\varepsilon)v^0$ is quasi-balanced. Although a game $v \in I^n$ is not necessarily quasi-balanced, we always have that the corresponding bargaining game v^1 is quasi-balanced because v^1 was already balanced due to $C(v^1) = I(v) \neq \emptyset$. So for any $v \in I^n$ we define the following critical value:

$$\varepsilon^{qb}(v) := \min\{\varepsilon;\ \varepsilon \geq 0, v^\varepsilon \in QB^n\}.$$

The critical value $\varepsilon^{qb}(v)$ is well defined since the subclass QB^n is a convex polyhedral cone. Clearly, $v \in QB^n$ iff $\varepsilon^{qb}(v) = 0$. Figs. 2 and 3 illustrate the geometric location of the least quasi-balanced ε-tax game $v^{\varepsilon^{qb}(v)}$ in the two possible cases.

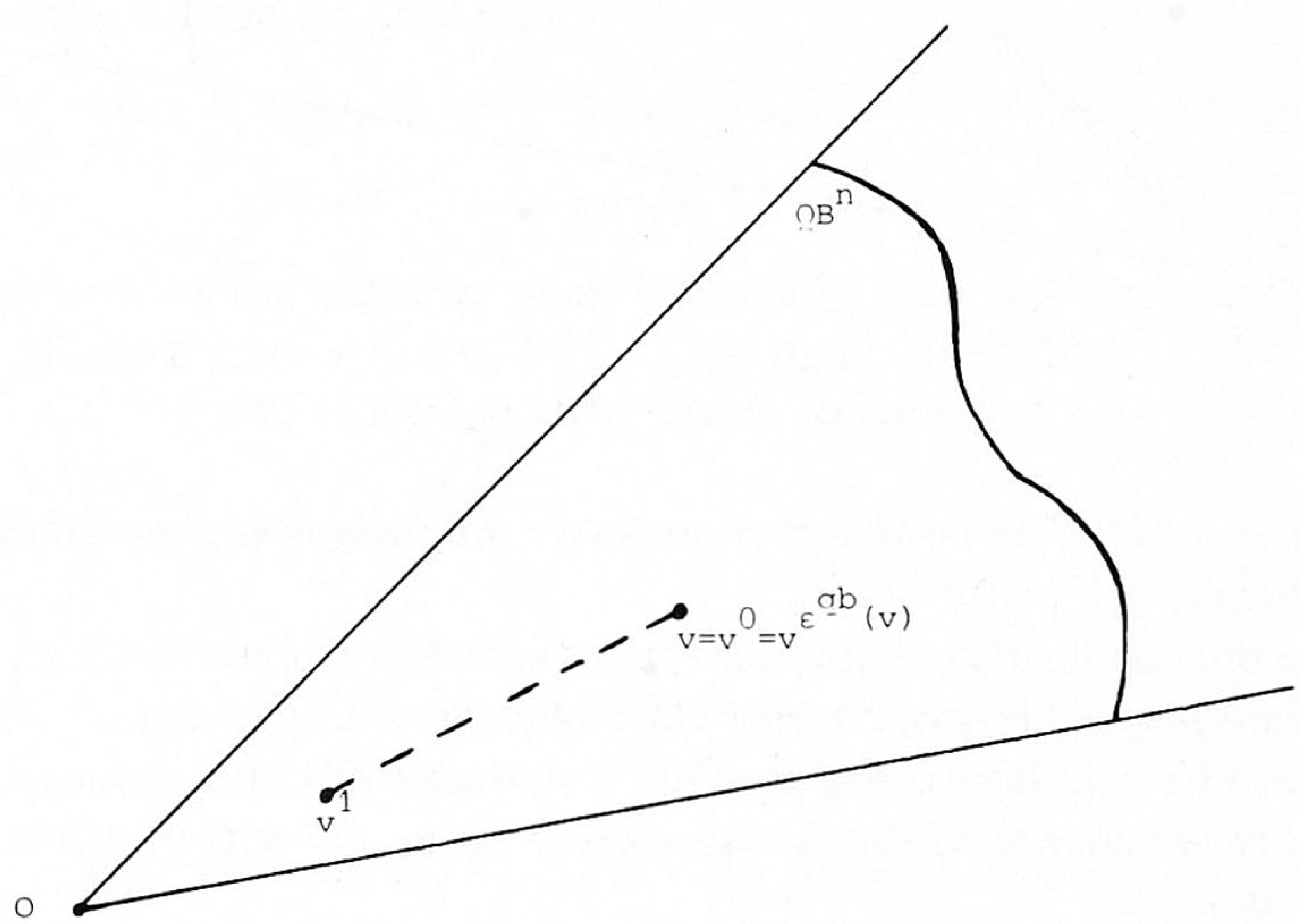

Fig. 2. The location of $v^{\varepsilon^{qb}(v)}$ in the case that $v \in QB^n$.

Definition. Let $v \in I^n$, where $n \geq 2$, and let $(N - D^v, w)$ be its corresponding dummy free game in the case $D^v \neq N, \emptyset$.

If $D^v = N, \emptyset$, then the *τ-value* $\tau(v)$ of the game v is defined as the τ-value of the least quasi-balanced ε-tax game $v^{\varepsilon^{qb}(v)} \in QB^n$.

If $D^v \neq N, \emptyset$, then the *τ-value* $\tau(v)$ of the game v is given by

$$\tau_i(v) = v(\{i\}) \quad \text{for all } i \in D^v$$

and

$$\tau_i(v) = \tau_i(w) \quad \text{for all } i \in N - D^v.$$

In view of the dummy player property and the dummy out property for the τ-value on QB^n, the τ-value on I^n as defined above can indeed be seen as an extension of the τ-value on QB^n. Properties of the extended τ-value are listed in the next theorem.

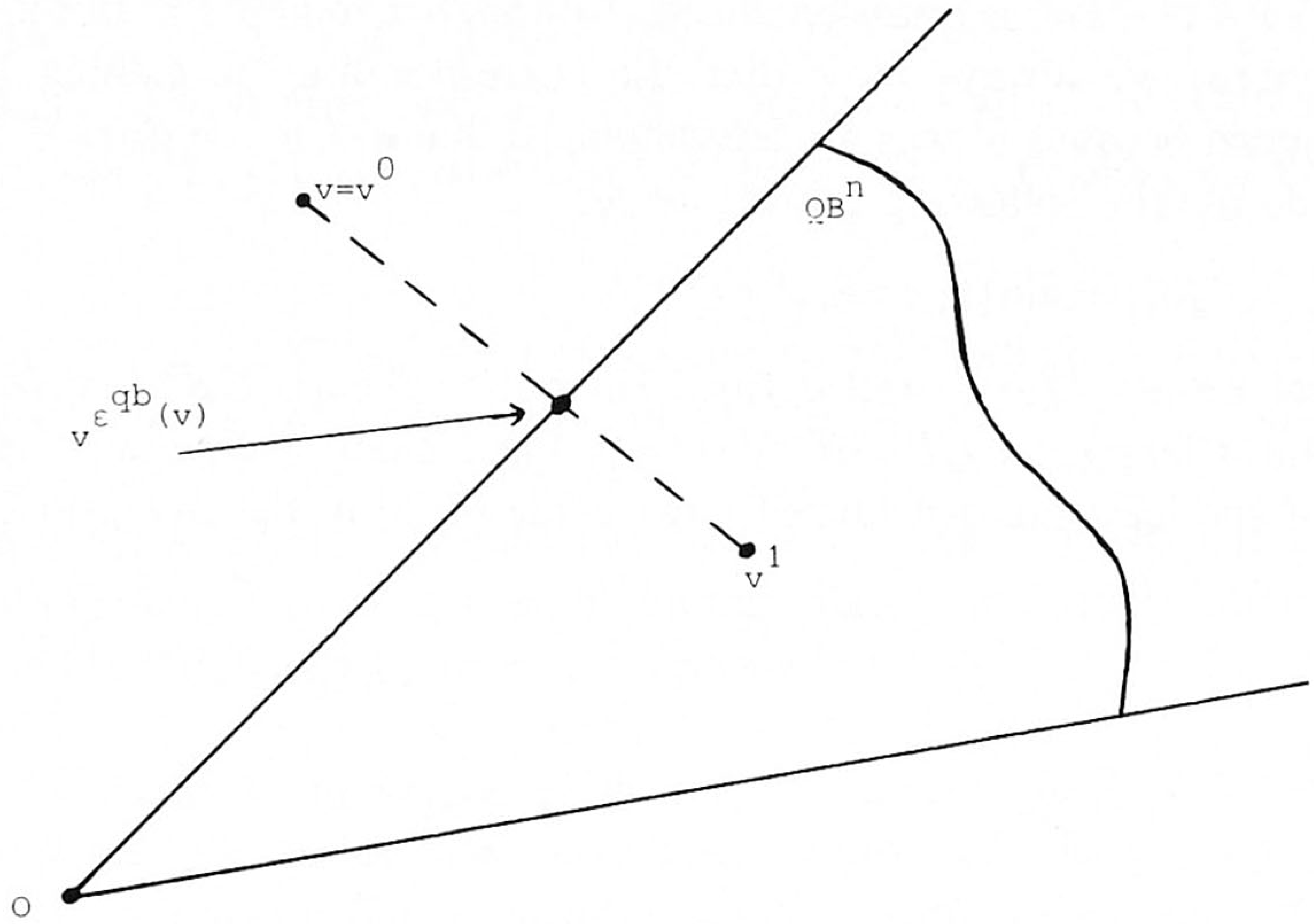

Fig. 3. The location of $v^{\varepsilon^{qb}(v)}$ in the case that $v \notin QB^n$.

Theorem. *Let $v \in I^n$. The τ-value $\tau(v)$ possesses the following properties:*

(i) *efficiency:* $\sum_{j=1}^n \tau_j(v) = v(N)$;
(ii) *individual rationality:* $\tau_i(v) \geq v(\{i\})$ *for all* $i \in N$;
(iii) *dummy player property:* $\tau_i(v) = v(\{i\})$ *for all* $i \in D^v$;
(iv) *symmetry:* $\tau_{\theta(i)}(\theta v) = \tau_i(v)$ *for all* $i \in N$ *and all* $\theta \in \Theta^n$;
(v) *relative invariance under S-equivalence:* $\tau(\alpha v + c) = \alpha\tau(v) + c$ *for all* $c \in \mathbb{R}^n$ *and all* $\alpha > 0$.

Proof. The case $n = 1$ is trivial. So we may assume that $n \geq 2$.

(i) If $D^v = \emptyset, N$, then it follows from the efficiency of the τ-value on QB^n that

$$\sum_{j=1}^n \tau_j(v) = \sum_{j=1}^n \tau_j(v^{\varepsilon^{qb}(v)}) = v^{\varepsilon^{qb}(v)}(N) = v(N). \tag{5.1}$$

If $D^v \neq \emptyset, N$, then

$$\sum_{j=1}^n \tau_j(v) = \sum_{j \in D^v} v(\{j\}) + \sum_{j \in N - D^v} \tau_j(w) = \sum_{j \in D^v} v(\{j\}) + w(N - D^v)$$
$$= \sum_{j \in D^v} v(\{j\}) + v(N - D^v) = v(N),$$

where the second equality follows from (5.1) because $D^w = \emptyset$.

(ii) Clearly, it is sufficient to consider the case $D^v = \emptyset, N$. Then it follows from the individual rationality of the τ-value on QB^n that

$$\tau_i(v) = \tau_i(v^{\varepsilon^{qb}(v)}) \geq v^{\varepsilon^{qb}(v)}(\{i\}) = v(\{i\}) \quad \text{for all } i \in N.$$

(iii) is trivial by noting that $\tau_i(v) = v(\{i\})$ for all $i \in N$ whenever $D^v = N$.

(iv) Since player i is a dummy in v if and only if player $\theta(i)$ is a dummy in θv, it is sufficient to consider the case $D^v = \emptyset, N$. Then it follows from $\varepsilon^{qb}(\theta v) = \varepsilon^{qb}(v)$, $(\theta v)^\varepsilon = \theta(v^\varepsilon)$ and the symmetry of the τ-value on QB^n that

$$\begin{aligned}\tau_{\theta(i)}(\theta v) &= \tau_{\theta(i)}((\theta v)^{\varepsilon^{qb}(\theta v)}) = \tau_{\theta(i)}((\theta v)^{\varepsilon^{qb}(v)}) = \tau_{\theta(i)}(\theta(v^{\varepsilon^{qb}(v)})) \\ &= \tau_i(v^{\varepsilon^{qb}(v)}) = \tau_i(v) \quad \text{for all } i \in N \text{ and all } \theta \in \Theta^n.\end{aligned}$$

(v) Let $\alpha > 0$ and $c \in \mathbb{R}^n$. Since player i is a dummy in $\alpha v + c$ iff player i is a dummy in v, it is sufficient to consider the case $D^v = \emptyset, N$. Then it follows from $\varepsilon^{qb}(\alpha v + c) = \varepsilon^{qb}(v)$, $(\alpha v + c)^\varepsilon = \alpha v^\varepsilon + c$ and the relative invariance under S-equivalence of the τ-value on QB^n that

$$\begin{aligned}\tau(\alpha v + c) &= \tau((\alpha v + c)^{\varepsilon^{qb}(\alpha v + c)}) = \tau((\alpha v + c)^{\varepsilon^{qb}(v)}) = \tau(\alpha v^{\varepsilon^{qb}(v)} + c) \\ &= \alpha\tau(v^{\varepsilon^{qb}(v)}) + c = \alpha\tau(v) + c. \qquad \square\end{aligned}$$

For the game v of the example given in Section 3 we have that $v \notin QB^3$, $D^v = \emptyset$, $\varepsilon^{qb}(v) = 1/10$ and hence $\tau(v) = \tau(v^{1/10}) = \{(18/10, 27/10, 90/10)\}$.

6. The stable core of a game

In the previous sections we considered the notions of the least (quasi-)balanced ε-tax game. In this section we consider the least convex ε-tax game in order to introduce the stable core of a game $v \in I^n$. Here a game $v \in G^n$ is said to be *convex* if

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T) \quad \text{for all } S, T \subset N.$$

Convex games were introduced in Shapley (1971) who proved that the core structure of a convex game is very regular. Furthermore, Shapley showed that the core of a convex game is the unique subset of the imputation set which satisfies an internal as well as an external stability condition. In other words, the core is the unique stable set of a convex game. For any game $v \in I^n$ it is straightforward to show that the corresponding bargaining game v^1 is convex. Since the subclass C^n of convex n-person games is a convex polyhedral cone, the critical value $\varepsilon^c(v)$ given by

$$\varepsilon^c(v) := \min\{\varepsilon;\ \varepsilon \geq 0, v^\varepsilon \text{ is convex}\}$$

is well defined. Clearly, v is convex iff $\varepsilon^c(v) = 0$. Furthermore, $\varepsilon^{qb}(v) \leq \varepsilon^b(v) \leq \varepsilon^c(v)$ since $C^n \subset B^n \subset QB^n$.

Definition. Let $v \in I^n$, where $n \geq 2$. The *stable core of the game* v is defined as the core of the least convex ε-tax game $v^{\varepsilon^c(v)} \in C^n$.

It follows from the lemma of Section 3 that if the relative reward to any coalition for its formation from the individuals is non-negative, then the least-tax-core is included in the stable core of the game.

Example. Let the 3-person game v be given by $v(\emptyset)=0=v(\{i\})$ for all $i \in N$ and $v(S)=1$ otherwise. Then $\varepsilon^{qb}(v)=\varepsilon^{b}(v)=1/3$, $\varepsilon^{c}(v)=1/2$ and $C(v^{1/3})=\{(1/3,1/3,1/3)\} \subset \operatorname{conv}\{(0,1/2,1/2),(1/2,0,1/2),(1/2,1/2,0)\}=C(v^{1/2})$.

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